

# EXISTENCE OF COMMON HYPERCYCLIC SUBSPACES FOR THE DERIVATIVE OPERATOR AND THE TRANSLATION OPERATORS

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**ABSTRACT.** We show that the non-zero multiples of the derivative operator and the non-zero multiples of non-trivial translation operators on the space of entire functions share a common hypercyclic subspace, *i.e.* a closed infinite-dimensional subspace in which each non-zero vector has a dense orbit for each of these operators.

## 1. INTRODUCTION

Let  $X$  be a separable infinite-dimensional Fréchet space and  $T$  a continuous and linear operator on  $X$ . We say that  $T$  is hypercyclic if there exists a vector  $x \in X$  (also called hypercyclic) such that the orbit  $\text{Orb}(x, T) := \{T^n x : n \geq 0\}$  is dense in  $X$ , and we denote by  $\text{HC}(T)$  the set of hypercyclic vectors for  $T$ .

If  $\text{HC}(T)$  is non-empty, it is known that  $\text{HC}(T)$  is a dense  $G_\delta$ -set [6] and that  $\text{HC}(T) \cup \{0\}$  contains a dense infinite-dimensional subspace [8, 12]. One can wonder if  $\text{HC}(T) \cup \{0\}$  is also spaceable, *i.e.* contains a closed infinite-dimensional subspace. It was proved in [16] that this is not the case in general: there exist some hypercyclic operators  $T$  for which  $\text{HC}(T) \cup \{0\}$  is not spaceable. If  $\text{HC}(T) \cup \{0\}$  is spaceable, we will say that  $T$  possesses a hypercyclic subspace. The interested reader can refer to two books [4, 11] for more information about hypercyclic operators and to the book [2] for more information about spaceability.

In this paper, we investigate the multiples of the derivative operator  $D$  on  $H(\mathbb{C})$  defined by  $Df = f'$  and the multiples of the translation operators  $T_a$  on  $H(\mathbb{C})$  defined by  $T_a f(\cdot) = f(\cdot + a)$ . Recall that  $H(\mathbb{C})$  is the space of entire functions on the whole complex plane endowed with the topology of uniform convergence on compact subsets which is induced by the sequence of norms  $(p_j)_{j \geq 1}$  given by

$$p_j(f) = \sup_{|z| < j} |f(z)|.$$

These operators are the first examples of hypercyclic operators [7, 14]. In fact, each non-zero multiple of the derivative operator [18, 15] and each non-zero multiple of a non-trivial translation operator [17] possesses a hypercyclic subspace. One can therefore wonder if a non-zero multiple of the derivative operator and a non-zero multiple of a non-trivial translation operator possess a common hypercyclic subspace, *i.e.* a closed infinite-dimensional subspace in which each non-zero vector is hypercyclic for each of these operators. A sufficient condition for the existence of a common hypercyclic subspace for a countable family can be found in [1].

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**Theorem 1.1** (Criterion  $M_0$  for countable families [1]). *Let  $X$  be a separable Fréchet space with a continuous norm and let  $\{T_\lambda\}_{\lambda \in \Lambda}$  be a family of operators where  $\Lambda$  is at most countable. Let  $M_0$  be a closed infinite-dimensional subspace of  $X$ . If for every  $\lambda \in \Lambda$ , there exists an increasing sequence  $(n_k)$  such that the operator  $T_\lambda$  satisfies the Hypercyclicity Criterion along  $(n_k)$  and such that for every  $x \in M_0$ ,  $T_\lambda^{n_k} x$  tends to 0, then the family  $\{T_\lambda\}_{\lambda \in \Lambda}$  possesses a common hypercyclic subspace.*

The difficulty in using this criterion relies on the existence of the subspace  $M_0$ . We will see in Section 2 how we can succeed to construct such a subspace for the operators  $\mu D$  and  $T_a$  in order to deduce the existence of a common hypercyclic subspace for these two operators.

Thanks to Costakis and Sambarino [10], we also know that the family of the non-zero multiples of the derivative operator  $\{\mu D\}_{\mu \neq 0}$  possesses a dense  $G_\delta$ -set of common hypercyclic vectors and that the family of non-trivial translation operators  $\{T_a\}_{a \in \mathbb{C} \setminus \{0\}}$  possesses a dense  $G_\delta$ -set of common hypercyclic vectors. In 2010, Shkarin [19] improved this last result by showing that the family  $\{\mu T_a\}_{a, \mu \in \mathbb{C} \setminus \{0\}}$  possesses a dense  $G_\delta$ -set of common hypercyclic vectors. The family  $\{\mu D\}_{\mu \neq 0} \cup \{\mu T_a\}_{a, \mu \in \mathbb{C} \setminus \{0\}}$  thus possesses a dense  $G_\delta$ -set of common hypercyclic vectors.

One can therefore wonder if the above families possess a common hypercyclic subspace. Since these families are uncountable, we cannot use the previous criterion. However, there also exist versions of criterion  $M_0$  for uncountable families parametrized by a one-dimensional set [3, 5]. In particular, it is shown in [5] that the uncountable family  $\{\mu D\}_{\mu \neq 0}$  possesses a common hypercyclic subspace. Unfortunately, we cannot apply these criteria to the family  $\{\mu T_a\}_{a, \mu \in \mathbb{C} \setminus \{0\}}$  and it is thus not known if the family  $\{\mu T_a\}_{a, \mu \in \mathbb{C} \setminus \{0\}}$  possesses a common hypercyclic subspace. However, we can easily adapt Theorem 4.1 in [5] in order to obtain a sufficient condition for the existence of common hypercyclic subspaces which can be applied to the two-dimensional family  $\{\mu T_a\}_{a, \mu \in \mathbb{C} \setminus \{0\}}$ . Indeed, if we look at the proof of this theorem, we can remark that we only need the following properties in order to construct a common hypercyclic subspace:

**Theorem 1.2** (Criterion  $M_0$  for uncountable families). *Let  $X$  be a Fréchet space with a continuous norm, let  $Y$  be a separable Fréchet space and let  $\{T_{k,\lambda}\}_{\lambda \in \Lambda}$  be a family of sequences of operators in  $L(X, Y)$ . Suppose that there exist chains  $(\Lambda_n^j)_{n \geq 1}$  of  $\Lambda$  ( $j = 0, 1, 2$ ) satisfying:*

- (i) *for every  $k, n \geq 1$ , the family  $\{T_{k,\lambda}\}_{\lambda \in \Lambda_n^0}$  is equicontinuous;*
- (ii) *for every  $n \geq 1$ , there exists a dense subset  $X_{n,0}$  of  $X$  such that for every  $x \in X_{n,0}$ ,*

$$T_{k,\lambda} x \xrightarrow[k \rightarrow \infty]{} 0 \quad \text{uniformly on } \lambda \in \Lambda_n^1;$$

- (iii) *for every  $l, n \geq 1$ , every  $\varepsilon > 0$ , every  $y \in Y$ , every  $K_0 \geq 0$ , there exist  $x \in X$  and  $K_1 \geq K_0$  such that  $p_l(x) < \varepsilon$  and such that for every  $\lambda \in \Lambda_n^2$ , there exists  $k \in [K_0, K_1]$  such that*

$$q_l(T_{k,\lambda} x - y) < \varepsilon$$

- (iv) *there exists an infinite-dimensional closed subspace  $M_0$  such that for any  $(\lambda, x) \in \Lambda \times M_0$ ,*

$$T_{k,\lambda} x \xrightarrow[k \rightarrow \infty]{} 0.$$

Then  $\{(T_{n,\lambda})_{n \geq 0}\}_{\lambda \in \Lambda}$  has a common hypercyclic subspace.

Thanks to this criterion, we will be able to show in Section 3 that the family  $\{\mu T_a\}_{a, \mu \in \mathbb{C} \setminus \{0\}}$  and even the family  $\{\mu D\}_{\mu \neq 0} \cup \{\mu T_a\}_{a, \mu \in \mathbb{C} \setminus \{0\}}$  possess a common hypercyclic subspace. This is the main result of this paper.

**Theorem 1.3.** *The family  $\{\lambda D\}_{\lambda \in \mathbb{C} \setminus \{0\}} \cup \{\mu T_a\}_{a, \mu \in \mathbb{C} \setminus \{0\}}$  possesses a common hypercyclic subspace.*

## 2. COMMON HYPERCYCLIC SUBSPACE FOR $\mu D$ AND $T_a$

Let  $\mu \neq 0$  and  $a \neq 0$ . We are interested in the existence of common hypercyclic subspaces for  $\mu D$  and  $T_a$ . Since the family  $\{\mu D, T_a\}$  is finite and since  $\mu D$  and  $T_a$  satisfy the Hypercyclicity Criterion along every increasing sequence, it follows from Theorem 1.1 that it suffices to find increasing sequences  $(n_k)$  and  $(m_k)$  and to find a closed infinite-dimensional  $M_0$  such that for every  $f \in H(\mathbb{C})$ , we have

$$(\mu D)^{n_k} f \xrightarrow[k \rightarrow \infty]{} 0 \quad \text{and} \quad T_a^{m_k} f \xrightarrow[k \rightarrow \infty]{} 0.$$

The existence of such increasing sequences  $(n_k)$  and  $(m_k)$  and such a closed infinite-dimensional subspace  $M_0$  will follow from the following theorem.

**Theorem 2.1** (Criterion  $(M_k)$  for finite families [5, Theorem 2.3]). *Let  $X$  be an infinite-dimensional Fréchet space with continuous norm, let  $Y$  be a separable Fréchet space and let  $\Lambda$  be a finite set. Let  $\{(T_{k,\lambda})_{k \geq 1}\}_{\lambda \in \Lambda}$  be a family of sequences of operators in  $L(X, Y)$ . Suppose that*

- (1) *there exists a dense subset  $X_0$  of  $X$  such that for any  $x \in X_0$ , any  $\lambda \in \Lambda$ ,*

$$T_{k,\lambda} x \xrightarrow[k \rightarrow \infty]{} 0;$$

- (2) *there exists a non-increasing sequence of infinite-dimensional closed subspaces  $(M_k)$  of  $X$  such that for every  $\lambda \in \Lambda$ , the sequence  $(T_{k,\lambda})_{k \geq 1}$  is equicontinuous along  $(M_k)$ .*

*Then for any map  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  there exist an increasing sequence of integers  $(k_s)_{s \geq 1}$  and an infinite-dimensional closed subspace  $M_0$  of  $X$  such that for any  $x \in M_0$ , any  $\lambda \in \Lambda$ ,*

$$T_{k,\lambda} x \xrightarrow[k \rightarrow \infty]{k \in I} 0,$$

*where  $I = \bigcup_{s \geq 1} [k_s, k_s + \phi(k_s)]$ .*

The proof of the existence of a common hypercyclic subspace for  $\mu D$  and  $T_a$  can then be divided into two steps:

- (1) There exists a dense set  $X_0$  and two increasing sequences of positive integers  $(n_k)$  and  $(m_k)$  such that for every  $f \in X_0$ ,  $(\mu D)^{n_k} f \rightarrow 0$  and  $T_a^{m_k} f \rightarrow 0$ .
- (2) There exists a non-increasing sequence of closed infinite-dimensional subspaces  $(M_k)$  such that for every  $j \geq 1$ , there exists a continuous norm  $q_j$  on  $H(\mathbb{C})$  such that for every  $k \geq j$ , every  $f \in M_k$ , we have

$$p_j((\mu D)^{n_k} f) \leq q_j(f) \quad \text{and} \quad p_j(T_a^{m_k} f) \leq q_j(f).$$

In order to construct the set  $X_0$  and the sequence  $(M_k)$ , we will repeatedly use the well-known Mergelyan's Theorem.

**Theorem 2.2** (Mergelyan's Theorem). *Let  $K$  be a compact subset of  $\mathbb{C}$  with connected complement. If  $h : K \rightarrow \mathbb{C}$  is a function continuous on  $K$  and holomorphic in the interior of  $K$  then for every  $\varepsilon > 0$ , there exists a polynomial  $P$  such that*

$$\sup_{z \in K} \|h(z) - P(z)\| < \varepsilon.$$

We notice that if  $K$  is a finite union of pairwise disjoint closed disks in  $\mathbb{C}$  then  $K$  is a compact subset with connected complement. In particular, we directly deduce the following lemma from Mergelyan's Theorem.

**Lemma 2.3.** *For every  $j \geq 1$ , there exists  $N \geq 0$  such that for every  $n \geq N$ , every  $\varepsilon > 0$ , every  $f \in H(\mathbb{C})$ , there exists a polynomial  $P$  such that*

$$p_j(P) < \varepsilon \quad \text{and} \quad p_j(T_a^n(P + f)) < \varepsilon.$$

*Proof.* Let  $j \geq 1$ . We consider  $N > \frac{2j}{|a|}$  so that for every  $n \geq N$ , we have  $D(0, j) \cap D(-an, j) = \emptyset$  where  $D(x, r) = \{z \in \mathbb{C} : |z - x| \leq r\}$ . We then obtain the desired result by applying Mergelyan's theorem to the function  $h$  defined from  $D(0, j) \cup D(-an, j)$  to  $\mathbb{C}$  by  $h(z) = -f(z)$  if  $z \in D(-an, j)$  and  $h(z) = 0$  otherwise.  $\square$

If  $X_0$  is the set of polynomials then it is obvious that  $(\mu D)^n P \rightarrow 0$  for every  $P \in X_0$ . However, if  $P$  is a non-zero polynomial, we never have  $T_a^n P \rightarrow 0$ , even if we look along a subsequence. Thanks to the above lemma, we can however perturb a dense sequence of polynomials in order to get the desired convergences.

**Proposition 2.4.** *There exist a dense set  $X_0$  and two increasing sequences of positive integers  $(n_k)$  and  $(m_k)$  such that for every  $f \in X_0$ ,  $(\mu D)^{n_k} f \rightarrow 0$  and  $T_a^{m_k} f \rightarrow 0$ .*

*Proof.* We consider a dense sequence of polynomials  $(P_{l,i})_{l \geq 0}$  and we construct by induction a family of polynomials  $(P_{l,k})_{l < k}$  such that the set  $X_0 = \{\sum_{i=l}^{\infty} P_{l,i} : l \geq 0\}$  is well-defined and satisfies the required conditions.

We start by letting  $n_1 = \deg(P_{0,0}) + 1$  and by using Lemma 2.3 in order to obtain a polynomial  $P_{0,1}$  and a positive integer  $m_1$  satisfying

- $p_1(P_{0,1}) < \frac{1}{2}$ ;
- $p_1((\mu D)^{n_1} P_{0,1}) < \frac{1}{2}$ ;
- $p_1(T_a^{m_1}(P_{0,0} + P_{0,1})) < \frac{1}{2}$ .

The second condition can be satisfied because the operator  $\mu D$  is continuous and there thus exists  $j \geq 1$  and  $\varepsilon > 0$  such that if  $p_j(P_{0,1}) < \varepsilon$  then  $p_1((\mu D)^{n_1} P_{0,1}) < \frac{1}{2}$ . More generally, if we assume that the family  $(P_{l,j})_{l < j}$  has been constructed for every  $j < k$  then we let  $n_k = \max\{\deg(P_{l,k-1}) : l \leq k-1\} + 1$  and we use Lemma 2.3 several times in order to obtain polynomials  $(P_{l,k})_{l < k}$  and a positive integer  $m_k > m_{k-1}$  such that for every  $l < k$

- $p_k(P_{l,k}) < \frac{1}{2^k}$ ;
- $p_k((\mu D)^{n_j} P_{l,k}) < \frac{1}{2^k}$  for every  $j \leq k$ ;
- $p_k(T_a^{m_j} P_{l,k}) < \frac{1}{2^k}$  for every  $j < k$ ;
- $p_k\left(T_a^{m_k}\left(\sum_{j=l}^k P_{l,j}\right)\right) < \frac{1}{2^k}$ .

The second and the third conditions can be satisfied by using as previously the continuity of  $\mu D$  and the continuity of  $T_a$ .

Let  $X_0 := \{\sum_{i=l}^{\infty} P_{l,i} : l \geq 0\}$ . We first remark that each series  $\sum_{i=l}^{\infty} P_{l,i}$  is convergent and that  $X_0$  is dense since for every  $i > l$ ,  $p_i(P_{l,i}) < \frac{1}{2^i}$ . Moreover, by definition of the sequence  $(n_j)$  we have  $D^{n_j} P_{l,i} = 0$  for every  $i < j$ . Therefore, if  $j \geq \max\{l, k\}$ , we have

$$\begin{aligned} p_k\left((\mu D)^{n_j}\left(\sum_{i=l}^{\infty} P_{l,i}\right)\right) &= p_k\left((\mu D)^{n_j}\left(\sum_{i=j}^{\infty} P_{l,i}\right)\right) \\ &\leq \sum_{i=j}^{\infty} p_i((\mu D)^{n_j} P_{l,i}) \leq \sum_{i=j}^{\infty} \frac{1}{2^i} \xrightarrow{j \rightarrow +\infty} 0 \end{aligned}$$

and

$$\begin{aligned} p_k\left(T_a^{m_j}\left(\sum_{i=l}^{\infty} P_{l,i}\right)\right) &\leq p_j\left(T_a^{m_j}\left(\sum_{i=l}^j P_{l,i}\right)\right) + \sum_{i=j+1}^{\infty} p_i(T_a^{m_j} P_{l,i}) \\ &< \frac{1}{2^j} + \sum_{i=j+1}^{\infty} \frac{1}{2^i} \xrightarrow{j \rightarrow +\infty} 0. \end{aligned}$$

□

Let  $p'_j(f) = \sum_{k=0}^{\infty} |x_k| j^k$  where  $f = \sum_{k=0}^{\infty} x_k z^k$ . The sequence  $(p'_j)$  is another increasing sequence of norms inducing the topology of  $H(\mathbb{C})$  which is particularly useful when we want to construct a hypercyclic subspace for the operators  $P(D)$  where  $P$  is a non-constant polynomial (see [15]). For instance, the existence of an increasing sequence  $(n_k)$  and a non-increasing sequence of closed infinite-dimensional subspaces  $(M_k)$  such that  $((\mu D)^{n_k})$  is equicontinuous along  $(M_k)$  relies on the fact that for every  $k$

$$\limsup_n \max_{m,j \leq k} \frac{p'_j((\mu D)^m z^n)}{p'_{2j}(z^n)} = \limsup_n \max_{m,j \leq k} \frac{|\mu|^m (\prod_{l=n-m+1}^n l) j^{n-m}}{(2j)^n} = 0.$$

Indeed, thanks to these equalities, we can find an increasing sequence  $(n_k)$  such that for every  $i, j \leq k$ ,

$$p'_j((\mu D)^{n_i} z^{n_k}) \leq p'_{2j}(z^{n_k})$$

and it is not difficult to prove that  $((\mu D)^{n_k})$  is then equicontinuous along  $(M_k)$  if we let  $M_k = \overline{\text{span}}\{z^{n_l} : l \geq k\}$ .

In fact, the construction of a convenient sequence  $(M_k)$  for  $\mu D$  mainly relies on the valuation of considered elements. The following lemma will allow us to adapt the above reasoning in order to obtain a non-increasing sequence of closed infinite-dimensional subspaces  $(M_k)$  working for both  $\mu D$  and  $T_a$ .

**Lemma 2.5.** *Let  $(m_k)$  be an increasing sequence of positive integers such that  $m_1|a| > 2$  and  $D(-m_j a, j) \cap D(-m_k a, k) = \emptyset$  for every  $j \neq k$ . Then for every  $k \geq 1$ , every  $\varepsilon > 0$ , every  $d \geq 0$ , there exists a polynomial  $P$  such that*

$$\text{val}(P) \geq d, \quad p_1(P) \geq \frac{1}{2} \quad \text{and} \quad p_j(T_a^{m_j} P) < \varepsilon \quad \text{for every } j \leq k.$$

*Proof.* We define a function  $g$  from  $D(0, 1) \cup \bigcup_{j \leq k} D(-am_j, j)$  to  $\mathbb{C}$  by  $g(z) = 1$  if  $z \in D(0, 1)$  and  $g(z) = 0$  otherwise. By assumption,  $g$  is well-defined and since  $g$  is continuous on  $D(0, 1) \cup \bigcup_{j \leq k} D(-am_j, j)$  and holomorphic on the interior of

$D(0, 1) \cup \bigcup_{j \leq k} D(-am_j, j)$ , we deduce from the Mergelyan's Theorem that there exists a polynomial  $Q$  such that

$$p_1(Q - 1) < \frac{1}{2} \quad \text{and} \quad \sup_{z \in D(-am_j, j)} |Q(z)| < \frac{\varepsilon}{(|a|m_j + j)^d} \quad \text{for every } j \leq k.$$

Let  $j \leq k$ . We deduce that if we let  $P(z) = z^d Q(z)$ , then we have  $\text{val}(P) \geq d$ ,

$$p_j(T_a^{m_j} P) = \sup_{z \in D(-am_j, j)} |P(z)| \leq (|a|m_j + j)^d \sup_{z \in D(-am_j, j)} |Q(z)| < \varepsilon$$

and since  $p_1(Q - 1) < \frac{1}{2}$ , we have

$$p_1(P) = \sup_{|z|=1} |z^d Q(z)| = \sup_{|z|=1} |Q(z)| \geq \frac{1}{2}.$$

□

**Proposition 2.6.** *Let  $(n_k)$  and  $(m_k)$  be two increasing sequences of positive integers. If  $m_1|a| > 2$  and  $D(-m_j a, j) \cap D(-m_k a, k) = \emptyset$  for every  $j \neq k$  then there exists a non-increasing sequence of closed infinite-dimensional subspaces  $(M_k)$  such that for every  $j \geq 1$ , there exists a continuous norm  $q_j$  on  $H(\mathbb{C})$  such that for every  $k \geq j$ , every  $f \in M_k$ , we have*

$$p_j((\mu D)^{n_k} f) \leq q_j(f) \quad \text{and} \quad p_j(T_a^{m_k} f) \leq q_j(f).$$

*Proof.* Since the sequence  $(p'_j)$  is an increasing sequence of norms inducing the topology of  $H(\mathbb{C})$ , we deduce that for every  $j \geq 0$ , there exists  $C_j > 0$  and  $l_j \geq 1$  such that for every  $f \in H(\mathbb{C})$ ,  $p_j(f) \leq C_j p'_{l_j}(f)$ .

Let  $P_0 = 1$ . We then consider a sequence  $(P_k)_{k \geq 1}$  such that for every  $k \geq 1$

- $\text{val}(P_k) > \deg(P_{k-1})$ ;
- $p_1(P_k) \geq \frac{1}{2}$ ;
- $\text{val}(P_k) \geq N_k$  where  $N_k$  satisfies

$$\sup_{n \geq N_k} \max_{i, j \leq k} \frac{p'_{l_j}((\mu D)^{n_i} e_n)}{p'_{2l_j}(e_n)} \leq 1.$$

- $p_j(T_a^{m_j} P_k) \leq \frac{1}{2^k}$  for every  $j \leq k$ .

The existence of  $N_k$  follows from the fact that

$$\begin{aligned} \limsup_n \max_{i, j \leq k} \frac{p'_{l_j}((\mu D)^{n_i} e_n)}{p'_{2l_j}(e_n)} &= \limsup_n \max_{i, j \leq k} \frac{l_j^{n-n_i} |\mu|^{n_i} \prod_{\nu=n-n_i+1}^n \nu}{(2l_j)^n} \\ &\leq \limsup_n \frac{\max\{1, |\mu|\}^{n_k} n^{n_k}}{2^n} = 0. \end{aligned}$$

and the existence of the sequence  $(P_k)_{k \geq 1}$  follows from Lemma 2.5.

Since  $\text{val}(P_{k+1}) > \deg(P_k)$ , the sequence  $(P_k)_{k \geq 1}$  is a basic sequence in  $H(\mathbb{C})$ . We let  $M_k = \overline{\text{span}}\{P_l : l \geq k\}$  and we show that this sequence of closed infinite-dimensional subspaces satisfies the required conditions. We first remark that if  $f = \sum_{l=k}^{\infty} a_l P_l$  is convergent then the sequence  $(a_l)_{l \geq k}$  is bounded by  $2C_1 p'_{l_1}(f)$ . Indeed, since  $p_1(P_l) \geq \frac{1}{2}$ , we have

$$|a_l| \leq 2p_1(a_l P_l) \leq 2C_1 p'_{l_1}(a_l P_l) \leq 2C_1 p'_{l_1}(f).$$

Let  $1 \leq j \leq k$  and  $f = \sum_{l=k}^{\infty} a_l P_l \in M_k$ . We have thus

$$p_j(T_a^{m_k} f) \leq \sum_{l=k}^{\infty} p_k(T_a^{m_k}(a_l P_l)) \leq \sum_{l=k}^{\infty} \frac{|a_l|}{2^l} \leq 2C_1 p'_{l_1}(f)$$

and

$$\begin{aligned} p_j((\mu D)^{n_k} f) &\leq C_j p'_{l_j}((\mu D)^{n_k} f) = \sum_{l=k}^{\infty} C_j p'_{l_j}((\mu D)^{n_k}(a_l P_l)) \\ &\leq \sum_{l=k}^{\infty} C_j p'_{2l_j}(a_l P_l) \quad \text{since } \text{val}(P_l) \geq N_l \\ &= C_j p'_{2l_j}(f). \end{aligned}$$

We have thus the desired result if we consider  $q_j = \max\{2C_1 p'_{l_1}, C_j p'_{2l_j}\}$ .  $\square$

**Theorem 2.7.** *The operators  $\mu D$  and  $T_a$  possess a common hypercyclic subspace.*

*Proof.* By Proposition 2.4, there exist a dense set  $X_0$  and two increasing sequences of positive integers  $(n_k)$  and  $(m_k)$  such that for every  $f \in X_0$ ,  $(\mu D)^{n_k} f \rightarrow 0$  and  $T_a^{m_k} f \rightarrow 0$ . Without loss of generality, we can assume that  $m_1|a| > 2$  and  $D(-m_j a, j) \cap D(-m_k a, k) \neq \emptyset$  for every  $j \neq k$ . Thanks to Proposition 2.6, we then get the existence of a non-increasing sequence of infinite-dimensional closed subspaces  $(M_k)$  of  $X$  such that  $((\mu D)^{n_k})_k$  and  $(T_a^{m_k})_k$  are equicontinuous along  $(M_k)$ . We now deduce from Theorem 2.1 applied with  $T_{k,1} = (\mu D)^{n_k}$  and  $T_{k,2} = T_a^{m_k}$  that there exist a closed infinite-dimensional closed subspace  $M_0$  and subsequences  $(n'_k)$  of  $(n_k)$  and  $(m'_k)$  of  $(m_k)$  such that for every  $f \in M_0$ ,

$$(\mu D)^{n'_k} f \rightarrow 0 \quad \text{and} \quad T_a^{m'_k} f \rightarrow 0.$$

Since  $\mu D$  and  $T_a$  satisfy the Hypercyclicity Criterion along each increasing sequence and thus in particular along  $(n'_k)$  and  $(m'_k)$ , the desired result follows from Theorem 1.1.  $\square$

### 3. COMMON HYPERCYCLIC SUBSPACE FOR $\{\mu D\}_{\mu \in \mathbb{C} \setminus \{0\}}$ AND $\{\mu T_a\}_{a, \mu \in \mathbb{C} \setminus \{0\}}$

In this section, we are interested in the family  $\{\mu D\}_{\mu \in \mathbb{C} \setminus \{0\}} \cup \{\mu T_a\}_{a, \mu \in \mathbb{C} \setminus \{0\}}$ . We remark that the considered family is now infinite and even uncountable. Fortunately, in order to show that the families  $\{\mu D\}_{\mu \in \mathbb{C} \setminus \{0\}}$  and  $\{\mu T_a\}_{a, \mu \in \mathbb{C} \setminus \{0\}}$  possess a common hypercyclic subspace, it suffices to show that the families  $\{e^b D\}_{b \in \mathbb{R}}$  and  $\{e^b T_a\}_{b \in \mathbb{R}, a \in \mathbb{T}}$  possess a common hypercyclic subspace since  $\text{HC}(\lambda T) = \text{HC}(T)$  if  $\lambda \in \mathbb{T}$  (see [13]) and since  $\text{HC}(T_a) = \text{HC}(T_b)$  if  $\frac{a}{b} \in \mathbb{R}^+$  (see [9]).

The existence of common hypercyclic subspaces for the families  $\{e^b D\}_{b \in \mathbb{R}}$  and  $\{e^b T_a\}_{b \in \mathbb{R}, a \in \mathbb{T}}$  will be obtained by applying Theorem 1.2 to the family  $\{T_{k,\lambda}\}_{\lambda \in \Lambda}$  given by

- $\Lambda := \mathbb{R} \cup (\mathbb{T} \times \mathbb{R})$ ,
- $T_{k,b} = e^b D^{n_k}$  for every  $b \in \mathbb{R}$ ,
- $T_{k,(a,b)} = e^b T_a^{m_k}$  for every  $a \in \mathbb{T} \setminus \{1\}$ , every  $b \in \mathbb{R}$ ,
- $T_{k,(1,b)} = e^b T_1^{m_k}$  for every  $b \in \mathbb{R}$

and by considering the chains  $(\Lambda_n)$  given by

$$\Lambda_n = [-n, n] \cup (\mathbb{T} \times [-n, n]) \quad \text{or} \quad [-n, n] \cup ((\mathbb{T} \setminus \{e^{i\theta} : |\theta| < \frac{1}{n}\}) \times [-n, n]) \cup (\{1\} \times [-n, n]).$$

The existence of a suitable subspace  $M_0$  will be again obtained thanks to Criterion  $(M_k)$ .

**Theorem 3.1** (Criterion  $(M_k)$  for infinite families [5, Theorem 2.3]). *Let  $X$  be an infinite-dimensional Fréchet space with continuous norm, let  $Y$  be a separable Fréchet space and let  $\Lambda$  be a set. Let  $\{(T_{k,\lambda})_{k \geq 1}\}_{\lambda \in \Lambda}$  be a family of sequences of operators in  $L(X, Y)$ . Suppose that there exist chains  $(\Lambda_n^j)_{n \geq 1}$  of  $\Lambda$  ( $j = 0, 1, 2$ ) satisfying:*

- (i) *for each  $n \in \mathbb{N}$  and each  $k \in \mathbb{N}$ , the family  $\{T_{k,\lambda}\}_{\lambda \in \Lambda_n^0}$  is equicontinuous;*
- (ii) *for each  $n \in \mathbb{N}$ , there exists a dense subset  $X_{n,0}$  of  $X$  such that for any  $x \in X_{n,0}$ ,*

$$T_{k,\lambda}x \xrightarrow[k \rightarrow \infty]{} 0 \quad \text{uniformly on } \lambda \in \Lambda_n^1;$$

- (iii) *there exists a non-increasing sequence of infinite-dimensional closed subspaces  $(M_k)$  of  $X$  such that for each  $n \geq 1$ , the family of sequences*

$$\{(T_{k,\lambda})_{k \geq 1}\}_{\lambda \in \Lambda_n^2}$$

*is uniformly equicontinuous along  $(M_k)$ .*

*Then for any map  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  there exist an increasing sequence of integers  $(k_s)_{s \geq 1}$  and an infinite-dimensional closed subspace  $M_0$  of  $X$  such that for any  $(x, \lambda) \in M_0 \times \Lambda$ ,*

$$T_{k,\lambda}x \xrightarrow[k \rightarrow \infty]{k \in I} 0,$$

*where  $I = \bigcup_{s \geq 1} [k_s, k_s + \phi(k_s)]$ .*

As previously, we will use repeatedly Mergelyan's Theorem in order to construct the set  $X_0$  and the sequence  $(M_k)$ . In view of our operators, we would like to consider the compact sets  $D(0, j) \cup \left[ \bigcup_{k=k_0}^{k_1} \bigcup_{a \in \mathbb{T}} D(-ak, j) \right]$ . Unfortunately, these sets have not a connected complement. For this reason, we will work with the following compact sets

$$D(0, j) \cup \left[ \bigcup_{k=k_0}^{k_1} \bigcup_{a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-\theta, \theta[ \}} D(-ak, j) \right] \cup \bigcup_{k=k_2}^{k_3} D(-k, j)$$

which have a connected complement if  $\theta > 0$  and  $k_0$  and  $k_2 - k_1$  are sufficiently big.

**Lemma 3.2.** *Let  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  be a map and let  $(m_k)$  and  $(t_k)$  be two increasing sequences. For every  $\theta \in ]0, \pi[$ , every  $j \geq 1$ , every  $K \geq 0$ , there exist  $k_1, k_2 \geq K$  such that for every  $\varepsilon > 0$ , every  $b > 0$ , every  $f \in H(\mathbb{C})$ , there exists a polynomial  $P$  such that*

- (1)  $p_j(P) < \varepsilon$ ;
- (2) *for every  $a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-\theta, \theta[ \}$ , every  $k \in [m_{k_1}, m_{k_1} + \phi(m_{k_1})]$ ,*

$$p_j(e^{bk} T_a^k(P + f)) < \varepsilon;$$

- (3) *for every  $k \in [t_{k_2}, t_{k_2} + \phi(t_{k_2})]$ ,*

$$p_j(e^{bk} T_1^k(P + f)) < \varepsilon.$$



*Proof.* Let  $\Omega_k := \bigcup_{a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-\theta, \theta[ \}} D(-ak, j)$ . There exists  $k_1 \geq K$  such that  $D(0, j) \cap \bigcup_{k=m_{k_1}}^{m_{k_1} + \phi(m_{k_1})} \Omega_k = \emptyset$  and such that the complement of  $D(0, j) \cup \bigcup_{k=m_{k_1}}^{m_{k_1} + \phi(m_{k_1})} \Omega_k$  is connected. We can also find  $k_2 \geq K$  such that  $\bigcup_{k=t_{k_2}}^{t_{k_2} + \phi(t_{k_2})} D(-k, j)$  is disjoint of  $\bigcup_{k=m_{k_1}}^{m_{k_1} + \phi(m_{k_1})} \Omega_k$  and of  $D(0, j)$ , and such that the complement of  $\Omega := D(0, j) \cup \bigcup_{k=m_{k_1}}^{m_{k_1} + \phi(m_{k_1})} \Omega_k \cup \bigcup_{k=t_{k_2}}^{t_{k_2} + \phi(t_{k_2})} D(-k, j)$  is connected.

We define a function  $h$  from  $\Omega$  to  $\mathbb{C}$  by  $h(z) = 0$  if  $z \in D(0, j)$  and  $h(z) = -f(z)$  otherwise. Since  $h$  is continuous on  $\Omega$  and holomorphic on the interior of  $\Omega$ , we deduce from the Mergelyan's Theorem that there exists a polynomial  $P$  such that

$$p_j(P) < \varepsilon \quad \text{and} \quad \sup_{z \in \Omega \setminus D(0, j)} |P(z) + f(z)| < \frac{\varepsilon}{e^{bC}},$$

where  $C = \max\{m_{k_1} + \phi(m_{k_1}), t_{k_2} + \phi(t_{k_2})\}$ . We deduce that

(1) for every  $a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-\theta, \theta[ \}$ , every  $k \in [m_{k_1}, m_{k_1} + \phi(m_{k_1})]$ ,

$$p_j(e^{bk} T_a^k(P + f)) \leq e^{bC} \sup_{z \in \Omega_k} |P(z) + f(z)| < \varepsilon;$$

(2) for every  $k \in [t_{k_2}, t_{k_2} + \phi(t_{k_2})]$ ,

$$p_j(e^{bk} T_1^k(P + f)) \leq e^{bC} \sup_{z \in D(-k, j)} |P(z) + f(z)| < \varepsilon;$$

□

Given a map  $\phi : \mathbb{N} \rightarrow \mathbb{N}$ , we say that a sequence  $(n_k)_{k \geq 1}$  is  $\phi$ -increasing if for every  $k \geq 1$ , we have

$$n_{k+1} > n_k + \phi(n_k).$$

Thanks to Lemma 3.2, we can then prove the following proposition which will allow us to show that the condition (ii) of Theorem 3.1 is satisfied by our family  $(T_k, \lambda)$ .

**Proposition 3.3.** *Let  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  and let  $(n_k)$ ,  $(m_k)$  and  $(t_k)$  be three  $\phi$ -increasing sequences. There exist a dense subset  $X_0$  of  $H(\mathbb{C})$  and three increasing sequences  $(k_{0,s})$ ,  $(k_{1,s})$  and  $(k_{2,s})$  such that for every  $b > 0$ , every  $\theta \in ]0, \pi[$ , every  $x \in X_0$ ,*

$$(1) \quad e^{bk} D^k x \xrightarrow[k \rightarrow \infty]{k \in \mathcal{N}} 0;$$

$$(2) \quad e^{bk} T_a^k x \xrightarrow[k \rightarrow \infty]{k \in \mathcal{M}} 0 \text{ uniformly on } a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-\theta, \theta[ \};$$

$$(3) \quad e^{bk} T_1^k x \xrightarrow[k \rightarrow \infty]{k \in \mathcal{T}} 0;$$

where  $\mathcal{N} = \bigcup_{s \geq 1} [n_{k_{0,s}}, n_{k_{0,s}} + \phi(n_{k_{0,s}})]$ ,  $\mathcal{M} = \bigcup_{s \geq 1} [m_{k_{1,s}}, m_{k_{1,s}} + \phi(m_{k_{1,s}})]$  and  $\mathcal{T} = \bigcup_{s \geq 1} [t_{k_{2,s}}, t_{k_{2,s}} + \phi(t_{k_{2,s}})]$ .

*Proof.* Let  $\phi$  be a map from  $\mathbb{N}$  to  $\mathbb{N}$  and let  $(n_k)$ ,  $(m_k)$  and  $(t_k)$  be three  $\phi$ -increasing sequences. We consider a dense sequence of polynomials  $(P_{l,i})_{l \geq 0}$  and we construct by induction three increasing sequences  $(k_{0,s})$ ,  $(k_{1,s})$  and  $(k_{2,s})$  and a family of polynomials  $(P_{l,k})_{l < k}$  such that the set  $X_0 = \{\sum_{i=l}^{\infty} P_{l,i} : l \geq 0\}$  is well-defined and satisfies the required conditions.

We start by choosing  $n_{k_{0,1}} > \deg(P_{0,0})$  and by using Lemma 3.2 in order to obtain a polynomial  $P_{0,1}$  and two positive integers  $k_{1,1}$  and  $k_{2,1}$  satisfying

- $p_1(P_{0,1}) < \frac{1}{2}$ ;
- $p_1(e^k D^k P_{0,1}) < \frac{1}{2}$  for every  $k \in [n_{k_{0,1}}, n_{k_{0,1}} + \phi(n_{k_{0,1}})]$ ;

- $p_1(e^k T_a^k(P_{0,0} + P_{0,1})) < \frac{1}{2}$  for every  $k \in [m_{k_{1,1}}, m_{k_{1,1}} + \phi(m_{k_{1,1}})]$  and every  $a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-1, 1[ \}$ ;
- $p_1(e^k T_1^k(P_{0,0} + P_{0,1})) < \frac{1}{2}$  for every  $k \in [t_{k_{2,1}}, t_{k_{2,1}} + \phi(t_{k_{2,1}})]$ ;

More generally, if we assume that the family  $(P_{l,j})_{l < j}$  has been constructed for every  $j < s$  then we choose  $n_{k_{0,s}} > \max\{\deg(P_{l,s-1}) : l \leq s-1\}$  with  $k_{0,s} > k_{0,s-1}$  and we use Lemma 3.2 several times in order to obtain polynomials  $(P_{l,s})_{l < s}$  and two positive integers  $k_{1,s} > k_{1,s-1}$  and  $k_{2,s} > k_{2,s-1}$  such that for every  $l < s$

- $p_s(P_{l,s}) < \frac{1}{2^s}$ ;
- $p_s(e^{sk} D^k P_{l,s}) < \frac{1}{2^s}$  for every  $k \in \bigcup_{j \leq s} [n_{k_{0,j}}, n_{k_{0,j}} + \phi(n_{k_{0,j}})]$ ;
- $p_s(e^{sk} T_a^k P_{l,s}) < \frac{1}{2^s}$  for every  $k \in \bigcup_{j < s} [m_{k_{1,j}}, m_{k_{1,j}} + \phi(m_{k_{1,j}})]$  and every  $a \in \mathbb{T}$ ;
- $p_s(e^{sk} T_1^k P_{l,s}) < \frac{1}{2^s}$  for every  $k \in \bigcup_{j < s} [t_{k_{2,j}}, t_{k_{2,j}} + \phi(t_{k_{2,j}})]$ ;
- $p_s\left(e^{sk} T_a^k \left(\sum_{j=l}^s P_{l,j}\right)\right) < \frac{1}{2^s}$  for every  $k \in [m_{k_{1,s}}, m_{k_{1,s}} + \phi(m_{k_{1,s}})]$  and every  $a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-\frac{1}{s}, \frac{1}{s}[ \}$ ;
- $p_s\left(e^{sk} T_1^k \left(\sum_{j=l}^s P_{l,j}\right)\right) < \frac{1}{2^s}$  for every  $k \in [t_{k_{2,s}}, t_{k_{2,s}} + \phi(t_{k_{2,s}})]$

Let  $X_0 := \{\sum_{i=l}^\infty P_{l,i} : l \geq 0\}$ . We first remark that each series  $\sum_{i=l}^\infty P_{l,i}$  is convergent and that  $X_0$  is dense since for every  $l < i$ ,  $p_i(P_{l,i}) < \frac{1}{2^i}$ . We can now prove the desired properties:

- (1) for every  $l, s \geq 1$ , every  $b > 0$ , every  $j \geq \max\{l, s, b\}$  and every  $k \in [n_{k_{0,j}}, n_{k_{0,j}} + \phi(n_{k_{0,j}})]$ ,

$$\begin{aligned} p_s\left(e^{bk} D^k \left(\sum_{i=l}^\infty P_{l,i}\right)\right) &= p_s\left(e^{bk} D^k \left(\sum_{i=j}^\infty P_{l,i}\right)\right) \\ &\leq \sum_{i=j}^\infty p_i(e^{ik} D^k P_{l,i}) \leq \sum_{i=j}^\infty \frac{1}{2^i} \rightarrow 0; \end{aligned}$$

- (2) for every  $l, s \geq 1$ , every  $b > 0$ , every  $\theta \in ]0, \pi[$ , every  $j \geq \max\{l, s, b, \frac{1}{\theta}\}$ , every  $k \in [m_{k_{1,j}}, m_{k_{1,j}} + \phi(m_{k_{1,j}})]$ , every  $a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-\theta, \theta[ \}$ , we have

$$\begin{aligned} p_s\left(e^{bk} T_a^k \left(\sum_{i=l}^\infty P_{l,i}\right)\right) &\leq p_j\left(e^{jk} T_a^k \left(\sum_{i=l}^j P_{l,i}\right)\right) + \sum_{i=j+1}^\infty p_i(e^{ik} T_a^k P_{l,i}) \\ &< \frac{1}{2^j} + \sum_{i=j+1}^\infty \frac{1}{2^i} \rightarrow 0. \end{aligned}$$

- (3) for every  $l, s \geq 1$ , every  $b > 0$ , every  $j \geq \max\{l, s, b\}$  and every  $k \in [t_{k_{2,j}}, t_{k_{2,j}} + \phi(t_{k_{2,j}})]$ ,

$$\begin{aligned} p_s\left(e^{bk} T_1^k \left(\sum_{i=l}^\infty P_{l,i}\right)\right) &\leq p_j\left(e^{jk} T_1^k \left(\sum_{i=l}^j P_{l,i}\right)\right) + \sum_{i=j+1}^\infty p_i(e^{ik} T_1^k P_{l,i}) \\ &< \frac{1}{2^j} + \sum_{i=j+1}^\infty \frac{1}{2^i} \rightarrow 0. \end{aligned}$$

□

We now need to show that the condition (iii) of Theorem 3.1 is also satisfied by our family  $(T_{k,\lambda})$ . In order to construct the required subspaces  $M_k$ , a control on the valuation of the polynomials given by Mergelyan's Theorem is necessary. We thus prove the following result.

**Lemma 3.4.** *Let  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  be a map and let  $(m_k)$  and  $(t_k)$  two  $\phi$ -increasing sequences. There exist two increasing sequences  $(k_{1,s})$  and  $(k_{2,s})$  such that for every  $b > 0$ , every  $\varepsilon > 0$ , every  $d \geq 0$ , every  $s \geq 1$ , there exists a polynomial  $P$  such that*

- (1)  $\text{val}(P) \geq d$  and  $p_1(P) \geq \frac{1}{2}$ ;
- (2) for every  $j \leq s$ , every  $a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-\frac{1}{s}, \frac{1}{s}[\}$ , every  $k \in [m_{k_{1,j}}, m_{k_{1,j}} + \phi(m_{k_{1,j}})]$ ,
$$p_j(e^{bk} T_a^k P) < \varepsilon;$$
- (3) for every  $j \leq s$ , every  $k \in [t_{k_{2,j}}, t_{k_{2,j}} + \phi(t_{k_{2,j}})]$ ,
$$p_j(e^{bk} T_1^k P) < \varepsilon.$$

*Proof.* Let  $\Omega_{k,s} = \bigcup_{a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-\frac{1}{s}, \frac{1}{s}[\}}$   $D(-ak, s)$ . We first remark that we can construct by induction two increasing sequences  $(k_{1,s})$  and  $(k_{2,s})$  such that the sets  $D(0, 1)$ ,  $\bigcup_{k=m_{k_{1,s}}}^{m_{k_{1,s}}+\phi(m_{k_{1,s}})} \Omega_{k,s}$  ( $s \geq 1$ ) and  $\bigcup_{k=t_{k_{2,s}}}^{t_{k_{2,s}}+\phi(t_{k_{2,s}})} D(-k, s)$  ( $s \geq 1$ ) are pairwise disjoint and such that for every  $s \geq 1$ , the complement of  $\Omega_s := D(0, 1) \cup \bigcup_{j \leq s} \bigcup_{k=m_{k_{1,j}}}^{m_{k_{1,j}}+\phi(m_{k_{1,j}})} \Omega_{k,j} \cup \bigcup_{j \leq s} \bigcup_{k=t_{k_{2,j}}}^{t_{k_{2,j}}+\phi(t_{k_{2,j}})} D(-k, j)$  is connected. Let  $b > 0$ ,  $\varepsilon > 0$ ,  $d \geq 0$  and  $s \geq 1$ . We can define a function  $h$  from  $\Omega_s$  to  $\mathbb{C}$  by  $h(z) = 1$  if  $z \in D(0, 1)$  and  $h(z) = 0$  otherwise. By assumption,  $h$  is well-defined and since  $h$  is continuous on  $\Omega_s$  and holomorphic on the interior of  $\Omega_s$ , we deduce from the Mergelyan's Theorem that there exists a polynomial  $Q$  such that

$$p_1(Q - 1) < \frac{1}{2} \quad \text{and} \quad \sup_{z \in \Omega_s \setminus D(0,1)} |Q(z)| < \frac{\varepsilon}{e^{bC} C^d},$$

where  $C = \sup\{|z| : z \in \Omega_s\}$ .

We deduce that if we let  $P(z) = z^d Q(z)$ , then

- (1)  $\text{val}(P) \geq d$  and
$$p_1(P) = \sup_{|z|=1} |z^d Q(z)| = \sup_{|z|=1} |Q(z)| \geq \frac{1}{2};$$
- (2) for every  $j \leq s$ , every  $a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-\frac{1}{j}, \frac{1}{j}[\}$ , every  $k \in [m_{k_{1,j}}, m_{k_{1,j}} + \phi(m_{k_{1,j}})]$ ,
$$p_j(e^{bk} T_a^k P) = e^{bk} \sup_{z \in D(-ak, j)} |z^d Q(z)| \leq e^{bC} C^d \sup_{z \in \Omega_{k,j}} |Q(z)| < \varepsilon;$$
- (3) for every  $j \leq s$ , every  $k \in [t_{k_{2,j}}, t_{k_{2,j}} + \phi(t_{k_{2,j}})]$ ,
$$p_j(e^{bk} T_1^k P) = e^{bk} \sup_{z \in D(-k, j)} |z^d Q(z)| \leq e^{bC} C^d \sup_{z \in D(-k, j)} |Q(z)| < \varepsilon.$$

□

**Proposition 3.5.** *Let  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  and let  $(n_k)$ ,  $(m_k)$  and  $(t_k)$  be  $\phi$ -increasing sequences. There exist increasing sequences  $(k_{1,s})$  and  $(k_{2,s})$ , and a non-increasing sequence of infinite-dimensional closed subspaces  $(M_s)$  of  $H(\mathbb{C})$  such that for every*

$b > 0$ , every  $\theta \in ]0, \pi[$ , every  $j \geq 1$ , there exists a continuous seminorm  $q$  of  $H(\mathbb{C})$  such that for every  $a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-\theta, \theta[ \}$ , every  $s \in \mathbb{N}$ , every  $x \in M_s$ , we have

- (1) for every  $k \in [n_s, n_s + \phi(n_s)]$ ,  $p_j(e^{bk} D^k x) \leq q(x)$ ;
- (2) for every  $k \in [m_{k_1, s}, m_{k_1, s} + \phi(m_{k_1, s})]$ ,  $p_j(e^{bk} T_a^k x) \leq q(x)$ ;
- (3) for every  $k \in [t_{k_2, s}, t_{k_2, s} + \phi(t_{k_2, s})]$ ,  $p_j(e^{bk} T_1^k x) \leq q(x)$ .

*Proof.* We recall that we let  $p'_j(f) = \sum_{k=0}^{\infty} |a_k| j^k$  where  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  and that for every  $j \geq 0$ , there exists  $C_j > 0$  and  $l_j$  such that for every  $f \in H(\mathbb{C})$ ,  $p_j(f) \leq C_j p'_{l_j}(f)$ .

Let  $P_0 = 1$  and let  $(k_{1, s})$  and  $(k_{2, s})$  be the sequences given by Lemma 3.4. We then consider a sequence  $(P_s)_{s \geq 1}$  of polynomials such that for every  $s \geq 1$

- $\text{val}(P_s) > \deg(P_{s-1})$  and  $p_1(P_s) \geq \frac{1}{2}$ ;
- $\text{val}(P_s) \geq N_s$  where  $N_s$  satisfies

$$\sup_{n \geq N_s} \max_{k \leq n_s + \phi(n_s)} \max_{j \leq s} \frac{p'_{l_j}(e^{s(n_s + \phi(n_s))} D^k e_n)}{p'_{l_j+1}(e_n)} \leq 2.$$

- $p_j(e^{sk} T_a^k P_s) \leq \frac{1}{2^s}$  for every  $j \leq s$ , every  $a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-\frac{1}{j}, \frac{1}{j}[ \}$  and every  $k \in [m_{k_1, j}, m_{k_1, j} + \phi(m_{k_1, j})]$ .
- $p_j(e^{sk} T_1^k P_s) < \frac{1}{2^s}$  for every  $j \leq s$ , every  $k \in [t_{k_2, j}, t_{k_2, j} + \phi(t_{k_2, j})]$ .

The existence of such a sequence of polynomials follows from the choice of sequences  $(k_{1, s})$  and  $(k_{2, s})$ .

Since  $\text{val}(P_{k+1}) > \deg(P_k)$ , the sequence  $(P_k)_{k \geq 1}$  is a basic sequence in  $H(\mathbb{C})$ . We let  $M_k = \overline{\text{span}}\{P_l : l \geq k\}$  and we show that this sequence of closed infinite-dimensional subspaces satisfies the required conditions. We first remark that if  $x = \sum_{l=k}^{\infty} a_l P_l$  is convergent then the sequence  $(a_l)$  is bounded by  $2C_1 p'_{l_1}(x)$  since

$$|a_l| \leq 2p_1(a_l P_l) \leq 2C_1 p'_{l_1}(a_l P_l) \leq 2C_1 p'_{l_1}(x).$$

- (1) Let  $b > 0$  and  $j \geq 1$ . For every  $s \geq \max\{b, j\}$ , every  $x = \sum_{l=s}^{\infty} a_l P_l \in M_s$ , we have for every  $k \in [n_s, n_s + \phi(n_s)]$ ,

$$\begin{aligned} p_j(e^{bk} D^k x) &\leq C_j p'_{l_j}(e^{bk} D^k x) \leq \sum_{l=s}^{\infty} C_j p'_{l_j}(e^{b(n_s + \phi(n_s))} D^k a_l P_l) \\ &\leq \sum_{l=s}^{\infty} 2C_j p'_{l_j+1}(a_l P_l) \quad \text{since } \text{val}(P_l) \geq N_l \\ &= 2C_j p'_{l_j+1}(x). \end{aligned}$$

and by continuity, there exists a continuous seminorm  $q_{0, b, j}$  such that for every  $x \in H(\mathbb{C})$ , every  $s < \max\{b, j\}$ , every  $k \in [n_s, n_s + \phi(n_s)]$

$$p_j(e^{bk} D^k x) \leq q_{0, b, j}(x).$$

We conclude that if we let  $q = \max\{q_{0, b, j}, 2C_j p'_{l_j+1}\}$ , then we have

$$p_j(e^{bn_s} D^{n_s} x) \leq q(x)$$

for every  $s \in \mathbb{N}$ , every  $x \in M_s$ .

- (2) Let  $b > 0$ ,  $\theta \in ]0, \pi[$  and  $j \geq 1$ . For every  $s \geq \max\{b, \frac{1}{\theta}, j\}$ , every  $x = \sum_{l=s}^{\infty} a_l P_l \in M_s$ , every  $a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-\theta, \theta[ \}$ , every  $k \in [m_{k_1, s}, m_{k_1, s} + \phi(m_{k_1, s})]$ , we have

$$p_j(e^{bk} T_a^k x) \leq \sum_{l=s}^{\infty} p_j(e^{bk} T_a^k(a_l P_l)) \leq \sum_{l=s}^{\infty} \frac{|a_l|}{2^l} \leq 2C_1 p'_{l_1}(x).$$

Moreover, by equicontinuity, there exists a continuous seminorm  $q_{1,b,j}$  such that for every  $x \in H(\mathbb{C})$ , every  $s < \max\{b, \frac{1}{\theta}, j\}$ , every  $a \in \mathbb{T}$ , every  $k \in [m_{k_1, s}, m_{k_1, s} + \phi(m_{k_1, s})]$ ,

$$p_j(e^{bk} T_a^k x) \leq q_{1,b,j}(x).$$

We conclude that if we let  $q = \max\{q_{1,b,j}, 2C_1 p'_{l_1}\}$ , then we have

$$p_j(e^{bk} T_a^k x) \leq q(x)$$

for every  $s \in \mathbb{N}$ , every  $x \in M_s$ , every  $k \in [m_{k_1, s}, m_{k_1, s} + \phi(m_{k_1, s})]$  and every  $a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-\theta, \theta[ \}$ .

- (3) Let  $b > 0$  and  $j \geq 1$ . For every  $s \geq \max\{b, j\}$ , every  $x = \sum_{l=s}^{\infty} a_l P_l \in M_s$ , every  $k \in [t_{k_2, s}, t_{k_2, s} + \phi(t_{k_2, s})]$ , we have

$$p_j(e^{bk} T_1^k x) \leq \sum_{l=s}^{\infty} p_j(e^{bk} T_1^k(a_l P_l)) \leq \sum_{l=s}^{\infty} \frac{|a_l|}{2^l} \leq 2C_1 p'_{l_1}(x).$$

and by continuity, there exists a continuous seminorm  $q_{2,b,j}$  such that for every  $x \in H(\mathbb{C})$ , every  $s < \max\{b, j\}$ , every  $k \in [t_{k_2, s}, t_{k_2, s} + \phi(t_{k_2, s})]$ ,

$$p_j(e^{bk} T_1^k x) \leq q_{2,b,j}(x).$$

We conclude that if we let  $q = \max\{q_{2,b,j}, 2C_1 p'_{l_1}\}$ , then we have

$$p_j(e^{bk} T_1^k x) \leq q(x)$$

for every  $s \in \mathbb{N}$ , every  $x \in M_s$ , every  $k \in [t_{k_2, s}, t_{k_2, s} + \phi(t_{k_2, s})]$ . □

In summary, by using Proposition 3.3 and Proposition 3.5, we are now able to show that our family  $(T_{k,\lambda})$  satisfies the conditions of Theorem 3.1. In other words, we can deduce that the condition (iv) of Theorem 1.2 is satisfied. Moreover, it follows from Proposition 3.3 that the condition (ii) of Theorem 1.2 is also satisfied. Since the equicontinuity will not be difficult to obtain, it remains to prove that the condition (iii) of Theorem 1.2 can be satisfied. This proof will follow from the following two lemmas used by Shkarin [19] to prove the existence of common hypercyclic vectors for the family  $\{\mu T_a\}_{\mu, a \in \mathbb{C} \setminus \{0\}}$

**Lemma 3.6** ([19, Lemma 3.4]). *For each  $\delta, C > 0$ , there is  $R > 0$  such that for any  $n \in \mathbb{N}$ , there exists a finite set  $S \subset \mathbb{C}$  such that  $|z| \in \mathbb{N}$  and  $nR + C \leq |z| \leq (n+1)R - C$  for any  $z \in S$ ,  $|z - z'| \geq C$  for every  $z, z' \in S$  with  $z \neq z'$  and for each  $w \in \mathbb{T}$ , there exists  $z \in S$  such that  $|w - \frac{z}{|z|}| < \frac{\delta}{|z|}$ .*

**Lemma 3.7** ([19, Lemma 3.5]). *Let  $f \in H(\mathbb{C})$  and  $l \geq 1$ . There exist  $m(l) \geq l$ ,  $C_l > 1$  and  $\delta > 0$  such that for every  $a \in \mathbb{T}$ , every  $b \in \mathbb{R}$ , every  $n \in \mathbb{N}$  and every  $g \in H(\mathbb{C})$ , if  $p_{m(l)}(f - e^{bn} T_a^n g) < \frac{1}{C_l}$  then  $p_l(f - e^{cn} T_w^n g) < 1$  for every  $c \in \mathbb{R}$  and every  $w \in \mathbb{T}$  satisfying  $|b - c| < \frac{\delta}{n}$  and  $|a - w| < \frac{\delta}{n}$ .*

**Proposition 3.8.** *There exists an increasing map  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  such that if  $(n_k)$ ,  $(m_k)$  and  $(t_k)$  are  $\phi$ -increasing sequences, then for every  $l, n \geq 1$ , every  $\varepsilon > 0$ , every  $f \in H(\mathbb{C})$ , every  $K \geq 1$ , there exist  $x \in H(\mathbb{C})$  and  $k_0, k_1, k_2 \geq K$  such that  $p_l(x) < \varepsilon$  and such that for every  $b \in [-n, n]$ , every  $a \in \mathbb{T}$ ,*

(1) *there exists  $s \in [n_{k_0}, n_{k_0} + \phi(n_{k_0})]$  such that*

$$p_l(e^{bs}D^s x - f) < \varepsilon;$$

(2) *there exists  $s \in [m_{k_1}, m_{k_1} + \phi(m_{k_1})]$  such that*

$$p_l(e^{bs}T_a^s x - f) < \varepsilon;$$

(3) *there exists  $s \in [t_{k_2}, t_{k_2} + \phi(t_{k_2})]$  such that*

$$p_l(e^{bs}T_1^s x - f) < \varepsilon;$$

*Proof.* We know that the family  $\{e^b D\}_{b \in [-n, n]}$  satisfies the Common Hypercyclicity Criterion for every  $n$ . In other words, for every  $l, n \geq 1$ , every  $\varepsilon > 0$ , every  $f \in H(\mathbb{C})$ , every  $N \geq 1$ , there exist  $x \in H(\mathbb{C})$  and  $N_0 \geq N$  such that  $p_l(x) < \varepsilon$  and such that for every  $b \in [-n, n]$ , there exists  $s \in [N, N_0]$  such that

$$p_l(e^{bs}D^s x - f) < \varepsilon;$$

If we consider a dense sequence  $(P_n)$  in  $H(\mathbb{C})$ , we can thus find an increasing sequence  $(\phi_n)$  such that for every  $n \geq 1$ , every  $k \leq n$ , there exists  $x \in H(\mathbb{C})$  such that  $p_n(x) < \frac{1}{n}$  and such that for every  $b \in [-n, n]$ , there exists  $s \in [n, n + \phi_n]$  such that

$$p_n(e^{bs}D^s x - P_k) < \frac{1}{n}.$$

In other words, for every  $l, n \geq 1$ , every  $\varepsilon > 0$ , every  $f \in H(\mathbb{C})$ , there exists  $M \geq 1$  such that for every  $m \geq M$ , there exists  $x \in H(\mathbb{C})$  such that  $p_l(x) < \varepsilon$  and such that for every  $b \in [-n, n]$ , there exists  $s \in [m, m + \phi_m]$  such that

$$p_l(e^{bs}D^s x - f) < \varepsilon.$$

Indeed, given  $l, n \geq 1, \varepsilon > 0$  and  $f \in H(\mathbb{C})$  it suffices to consider  $M \geq \max\{l, n, \frac{2}{\varepsilon}, n_f\}$  where  $n_f$  satisfies  $p_l(f - P_{n_f}) < \frac{\varepsilon}{2}$ .

We consider an increasing map  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  satisfying  $\phi(n) \geq \phi_n$  for every  $n$  and  $\sum_{m=k}^{k+\phi(k)} \frac{1}{m} \rightarrow \infty$ . Let  $(n_k)$ ,  $(m_k)$  and  $(t_k)$  be  $\phi$ -increasing sequences and let  $l, n \geq 1, \varepsilon > 0, f \in H(\mathbb{C})$  and  $K \geq 1$ . We use Lemma 3.6 and Lemma 3.7 in order to determine a set of pairwise disjoint closed disks allowing us to get the desired approximations for the translation operators.

By Lemma 3.7, we deduce that there exist an integer  $m(l) \geq l$  and real numbers  $C_l > 1$  and  $\delta > 0$  such that for every  $a \in \mathbb{T}$ , every  $b \in \mathbb{R}$ , every  $m \in \mathbb{N}$  and every  $g \in H(\mathbb{C})$ , if  $p_{m(l)}(f - e^{bm}T_a^m g) < \frac{1}{C_l}$  then  $p_l(f - e^{cm}T_w^m g) < 1$  for every  $c \in \mathbb{R}$ ,  $w \in \mathbb{T}$  satisfying  $|b - c| < \frac{\delta}{m}$  and  $|a - w| < \frac{\delta}{m}$ . By using Lemma 3.6 with  $C = 4m(l)$ , we then obtain  $R \geq 1$  and finite sets  $S_m$  such that  $|z| \in \mathbb{N}$  and  $mR + 4m(l) \leq |z| \leq (m+1)R - 4m(l)$  for any  $z \in S_m$ ,  $|z - z'| \geq 4m(l)$  for every  $z, z' \in S_m$  with  $z \neq z'$ , and for each  $w \in \mathbb{T}$ , there exists  $z \in S_m$  such that  $|w - \frac{z}{|z|}| < \frac{\delta}{|z|}$ .

Let  $i_k$  and  $I_k \in \mathbb{N}$  satisfying

$$[Ri_k, R(i_k + I_k)] \subset [m_k, m_k + \phi(m_k)] \subset [R(i_k - 1), R(i_k + I_k + 1)].$$

Since  $\sum_{m=k}^{k+\phi(k)} \frac{1}{m} \rightarrow \infty$ , there exists  $k_1 \geq K$  such that  $\sum_{m=i_{k_1}}^{i_{k_1}+I_{k_1}-1} \frac{\delta R^{-1}}{m+1} > 2n$ . Indeed, we have

$$\begin{aligned} \sum_{m=i_k}^{i_k+I_k-1} \frac{\delta R^{-1}}{m+1} &\geq \sum_{m=\lfloor \frac{m_k}{R} \rfloor + 2}^{\lceil \frac{m_k+\phi(m_k)}{R} \rceil - 1} \frac{\delta R^{-1}}{m} \\ &\geq \sum_{m=\lfloor \frac{m_k}{R} \rfloor}^{\lfloor \frac{m_k}{R} \rfloor + \lceil \frac{\phi(m_k)}{R} \rceil} \frac{\delta R^{-1}}{m} - 3 \frac{\delta}{m_k - 1} \\ &\geq \sum_{m=m_k}^{m_k + \lceil \frac{\phi(m_k)}{R} \rceil} \frac{\delta R^{-1}}{m} - 3 \frac{\delta}{m_k - 1} \\ &\geq \frac{1}{R} \sum_{m=m_k}^{m_k + \phi(m_k)} \frac{\delta R^{-1}}{m} - 3 \frac{\delta}{m_k - 1} \rightarrow \infty \end{aligned}$$

With the same arguments, we obtain the existence of  $k_2$ ,  $j_{k_2}$  and  $J_{k_2}$  with  $j_{k_2} > i_{k_1} + I_{k_1}$  such that

$$[Rj_{k_2}, R(j_{k_2} + J_{k_2})] \subset [t_{k_2}, t_{k_2} + \phi(t_{k_2})] \quad \text{and} \quad \sum_{m=j_{k_2}}^{j_{k_2}+J_{k_2}-1} \frac{\delta R^{-1}}{m+1} > 2n.$$

Since  $\sum_{m=i_{k_1}}^{i_{k_1}+I_{k_1}-1} \frac{\delta R^{-1}}{m+1} > 2n$ , we can now pick  $b_1, \dots, b_{I_{k_1}} \in [-n, n]$  such that  $[-n, n] \subset \bigcup_{k=1}^{I_{k_1}} [b_k - \frac{\delta R^{-1}}{(i_{k_1}+k)}, b_k + \frac{\delta R^{-1}}{(i_{k_1}+k)}]$  and we can pick  $c_1, \dots, c_{J_{k_2}} \in [-n, n]$  such that  $[-n, n] \subset \bigcup_{k=1}^{J_{k_2}} [c_k - \frac{\delta R^{-1}}{(j_{k_2}+k)}, c_k + \frac{\delta R^{-1}}{(j_{k_2}+k)}]$ .

Let  $S = \bigcup_{k=1}^{I_{k_1}} S_{i_{k_1}+k-1} \cup \bigcup_{k=1}^{J_{k_2}} S_{j_{k_2}+k-1}$  and  $\Lambda = S \cup \{0\}$ . By definition of sets  $S_m$ , we have  $|z - u| \geq 4m(l)$  for every  $z, u \in \Lambda$ . It follows that  $\bigcup_{z \in \Lambda} D(z, m(l))$  is a compact set with connected complement. Let  $N = \max\{|z| : z \in \Lambda\}$ . We can then deduce from Mergelyan's Theorem that there exists a polynomial  $P$  such that  $p_{m(l)}(P) < \frac{\varepsilon}{2}$  and such that

$$p_{m(l)}(T_z P - e^{-b_k|z|} f) < \frac{\varepsilon}{2C_l} e^{-nN} \quad \text{for each } k \in [1, I_{k_1}], \text{ each } z \in S_{i_{k_1}+k-1}$$

and

$$p_{m(l)}(T_z P - e^{-c_k|z|} f) < \frac{\varepsilon}{2C_l} e^{-nN} \quad \text{for each } k \in [1, J_{k_2}], \text{ each } z \in S_{j_{k_2}+k-1}.$$

We deduce that  $p_l(P) < \frac{\varepsilon}{2}$ ,

$$p_{m(l)}(e^{b_k|z|} T_{\frac{z}{|z|}} P - f) < \frac{\varepsilon}{2C_l} \quad \text{for each } k \in [1, I_{k_1}], \text{ each } z \in S_{i_{k_1}+k-1}$$

and

$$p_{m(l)}(e^{c_k|z|} T_{\frac{z}{|z|}} P - f) < \frac{\varepsilon}{2C_l} \quad \text{for each } k \in [1, J_{k_2}], \text{ each } z \in S_{j_{k_2}+k-1}.$$

Let  $b \in [-n, n]$  and  $a \in \mathbb{T}$ . There exists  $k \in [1, I_{k_1}]$  such that  $|b - b_k| < \frac{\delta R^{-1}}{(i_{k_1}+k)}$ . By definition of sets  $S_m$ , we can also find  $z \in S_{i_{k_1}+k-1}$  such that  $|a - \frac{z}{|z|}| < \frac{\delta}{|z|}$ . Since

$|z| < R(i_{k_1} + k)$ , we have  $|b - b_k| < \frac{\delta}{|z|}$  and thus by definition of  $m(l)$

$$p_l(e^{b|z|} T_a^{|z|} P - f) < \frac{\varepsilon}{2}.$$

In particular, since  $|z| \in [Ri_k, R(i_k + I_k)] \subset [m_{k_1}, m_{k_1} + \phi(m_{k_1})]$ , we conclude that for every  $b \in [-n, n]$ , every  $a \in \mathbb{T}$ , there exists  $s \in [m_{k_1}, m_{k_1} + \phi(m_{k_1})]$  such that

$$p_l(e^{bs} T_a^s P - f) < \frac{\varepsilon}{2}.$$

Similarly, we deduce that for every  $b \in [-n, n]$ , every  $a \in \mathbb{T}$ , there exists  $s \in [t_{k_2}, t_{k_2} + \phi(t_{k_2})]$  such that  $p_l(e^{bs} T_a^s P - f) < \frac{\varepsilon}{2}$  and we have thus in particular

$$p_l(e^{bs} T_1^s P - f) < \frac{\varepsilon}{2}.$$

Finally, we deduce from the equicontinuity that there exists  $L \geq l$  and  $C > 0$  such that for every polynomial  $Q$  if  $p_L(Q) < C$  then for every  $b \in [-n, n]$ , every  $a \in \mathbb{T}$ , there exists  $s \in [m_{k_1}, m_{k_1} + \phi(m_{k_1})]$  such that

$$p_l(e^{bs} T_a^s (P + Q) - f) < \varepsilon$$

and there exists  $s \in [t_{k_2}, t_{k_2} + \phi(t_{k_2})]$  such that

$$p_l(e^{bs} T_1^s (P + Q) - f) < \varepsilon.$$

Moreover, thanks to our choice of  $\phi$ , we know that there exists  $M \geq 1$  such that for every  $m \geq M$ , there exists  $x \in H(\mathbb{C})$  such that  $p_l(x) < \varepsilon$ , such that  $p_L(x) < C$  and such that for every  $b \in [-n, n]$ , there exists  $s \in [m, m + \phi(m)]$  satisfying

$$p_l(e^{bs} D^s x - f) < \varepsilon.$$

In particular, if we consider  $k_0 \geq K$  such that  $n_{k_0} \geq \max\{M, \deg P\}$ , there exists  $Q \in H(\mathbb{C})$  such that  $p_l(Q) < \varepsilon$ , such that  $p_L(Q) < C$  and such that for every  $b \in [-n, n]$ , there exists  $s \in [n_{k_0}, n_{k_0} + \phi(n_{k_0})]$  such that

$$p_l(e^{bs} D^s (P + Q) - f) < \varepsilon \quad \text{since } D^s P = 0.$$

Since  $p_L(Q) < C$ , the vector  $x := P + Q$  then satisfies all desired properties.  $\square$

We are now able to prove the existence of common hypercyclic subspaces for the family  $\{\lambda D\}_{\lambda \in \mathbb{C} \setminus \{0\}} \cup \{\mu T_a\}_{a, \mu \in \mathbb{C} \setminus \{0\}}$ .

*Proof of Theorem 1.3.* Let  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  be the map given by Proposition 3.8. We use Proposition 3.3 with  $\phi$  in order to obtain the existence of a dense subset  $X_0$  of  $H(\mathbb{C})$  and three increasing sequences  $(n_k)$ ,  $(m_k)$  and  $(t_k)$  such that for every  $b > 0$ , every  $\theta \in ]0, \pi[$ , every  $x \in X_0$ ,

- (1)  $e^{bk} D^k x \xrightarrow[k \rightarrow \infty]{k \in \mathcal{N}} 0$ ;
- (2)  $e^{bk} T_a^k x \xrightarrow[k \rightarrow \infty]{k \in \mathcal{M}} 0$  uniformly on  $a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-\theta, \theta[ \}$ ;
- (3)  $e^{bk} T_1^k x \xrightarrow[k \rightarrow \infty]{k \in \mathcal{T}} 0$ ;

where  $\mathcal{N} = \bigcup_{k \geq 1} [n_k, n_k + \phi(n_k)]$ ,  $\mathcal{M} = \bigcup_{k \geq 1} [m_k, m_k + \phi(m_k)]$  and  $\mathcal{T} = \bigcup_{k \geq 1} [t_k, t_k + \phi(t_k)]$ .

We then use Proposition 3.5 in order to obtain increasing sequences  $(k_{1,s})$  and  $(k_{2,s})$ , and a non-increasing sequence of infinite-dimensional closed subspaces  $(M_s)$  of  $H(\mathbb{C})$  such that for every  $b > 0$ , every  $\theta \in ]0, \pi[$ , every  $j \geq 1$ , there exists a



continuous seminorm  $q$  of  $H(\mathbb{C})$  such that for every  $a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-\theta, \theta[ \}$ , every  $s \in \mathbb{N}$ , every  $x \in M_s$ , we have

- (1) for every  $k \in [n_s, n_s + \phi(n_s)]$ ,  $p_j(e^{bk} D^k x) \leq q(x)$ ;
- (2) for every  $k \in [m_{k_1, s}, m_{k_1, s} + \phi(m_{k_1, s})]$ ,  $p_j(e^{bk} T_a^k x) \leq q(x)$ ;
- (3) for every  $k \in [t_{k_2, s}, t_{k_2, s} + \phi(t_{k_2, s})]$ ,  $p_j(e^{bk} T_1^k x) \leq q(x)$ .

Let  $(\tilde{n}_k)$  be the increasing enumeration of  $\bigcup_{s \geq 1} [n_s, n_s + \phi(n_s)]$ , let  $(\tilde{m}_k)$  be the increasing enumeration of  $\bigcup_{s \geq 1} [m_{k_1, s}, m_{k_1, s} + \phi(m_{k_1, s})]$  and let  $(\tilde{t}_k)$  be the increasing enumeration of  $\bigcup_{s \geq 1} [t_{k_2, s}, t_{k_2, s} + \phi(t_{k_2, s})]$ . We consider the family  $\{(T_{k, \lambda})_{k \geq 1}\}_{\lambda \in \Lambda}$  where

- $\Lambda := \mathbb{R} \cup (\mathbb{T} \times \mathbb{R})$ ,
- $T_{k, b} = e^b D^{\tilde{n}_k}$  for every  $b \in \mathbb{R}$ ,
- $T_{k, (a, b)} = e^b T_a^{\tilde{m}_k}$  for every  $a \in \mathbb{T} \setminus \{1\}$ , every  $b \in \mathbb{R}$
- $T_{k, (1, b)} = e^b T_1^{\tilde{t}_k}$  for every  $b \in \mathbb{R}$ ,

and we show that each assumption of Theorem 3.1 is satisfied for this family.

- (i) Since the family  $(T_a)_{a \in \mathbb{T}}$  is equicontinuous, we easily deduce that for every  $k, n \geq 1$ , the family  $\{T_{k, \lambda}\}_{\lambda \in [-n, n] \cup (\mathbb{T} \setminus \{1\}) \times [-n, n]}$  is equicontinuous;
- (ii) Let  $\Lambda_n = [-n, n] \cup ((\mathbb{T} \setminus \{e^{i\theta} : |\theta| < \frac{1}{n}\}) \times [-n, n]) \cup (\{1\} \times [-n, n])$ . By definition of  $X_0$ , we can deduce that for every  $n \geq 1$ , for every  $x \in X_0$ ,

$$T_{k, \lambda} x \xrightarrow[k \rightarrow \infty]{} 0 \quad \text{uniformly on } \Lambda_n;$$

- (iii) We let  $s_k = \min\{s \in \mathbb{N} : \tilde{n}_k \leq n_s + \phi(n_s), \tilde{m}_k \leq m_{k_1, s} + \phi(m_{k_1, s}) \text{ and } \tilde{t}_k \leq t_{k_2, s} + \phi(t_{k_2, s})\}$  and  $\tilde{M}_k := M_{s_k}$ . For each  $n \geq 1$ , we can then deduce that the family  $\{(T_{k, \lambda})_{k \geq 1}\}_{\lambda \in \Lambda_n}$  is uniformly equicontinuous along  $(\tilde{M}_k)$ , i.e. for every  $j \geq 1$ , there exists a continuous seminorm  $q$  of  $X$  so that for every  $k$ , every  $x \in \tilde{M}_k$ ,

$$\sup_{\lambda \in \Lambda_n} p_j(T_{k, \lambda} x) \leq q(x).$$

Indeed, for every  $n, j \geq 1$ , we know that there exists a continuous seminorm  $q$  of  $H(\mathbb{C})$  such that for every  $a \in \mathbb{T} \setminus \{e^{i\alpha} : \alpha \in ]-\frac{1}{n}, \frac{1}{n}[\}$ , every  $s \in \mathbb{N}$ , every  $x \in M_s$ , we have

- (a) for every  $k \in [n_s, n_s + \phi(n_s)]$ ,  $p_j(e^{nk} D^k x) \leq q(x)$ ;
- (b) for every  $k \in [m_{k_1, s}, m_{k_1, s} + \phi(m_{k_1, s})]$ ,  $p_j(e^{nk} T_a^k x) \leq q(x)$ ;
- (c) for every  $k \in [t_{k_2, s}, t_{k_2, s} + \phi(t_{k_2, s})]$ ,  $p_j(e^{nk} T_1^k x) \leq q(x)$ .

Therefore, for every  $k \geq 1$  and every  $x \in \tilde{M}_k$ , if we assume that  $\tilde{n}_k \in [n_{s(0)}, n_{s(0)} + \phi(n_{s(0)})]$ , that  $\tilde{m}_k \in [m_{k_1, s(1)}, m_{k_1, s(1)} + \phi(m_{k_1, s(1)})]$  and that  $\tilde{t}_k \in [t_{k_2, s(2)}, t_{k_2, s(2)} + \phi(t_{k_2, s(2)})]$ , we have  $x \in M_{s(0)} \cap M_{s(1)} \cap M_{s(1)}$  and thus

$$\sup_{\lambda \in \Lambda_n} p_j(T_{k, \lambda} x) \leq q(x).$$

We deduce from Theorem 3.1 that for every map  $\tilde{\phi} : \mathbb{N} \rightarrow \mathbb{N}$ , there exist an increasing sequence of integers  $(l_i)_{i \geq 1}$  and an infinite-dimensional closed subspace  $M_0$  of  $H(\mathbb{C})$  such that for any  $(x, \lambda) \in M_0 \times \Lambda$ ,

$$T_{k, \lambda} x \xrightarrow[k \rightarrow \infty]{k \in I} 0,$$

where  $I = \bigcup_{i \geq 1} [l_i, l_i + \tilde{\phi}(l_i)]$ . In particular, we can choose  $\tilde{\phi}$  such that for every increasing sequence of integers  $(l_i)_{i \geq 1}$ , we have  $\{\tilde{n}_k : k \in I\} \supset \bigcup_s [n_{K_0, s}, n_{K_0, s} + \phi(n_{K_0, s})]$ ,  $\{\tilde{m}_k : k \in I\} \supset \bigcup_s [m_{K_1, s}, m_{K_1, s} + \phi(m_{K_1, s})]$  and  $\{\tilde{t}_k : k \in I\} \supset$

$\bigcup_s [t_{K_{2,s}}, t_{K_{2,s}} + \phi(t_{K_{2,s}})]$  for some increasing sequence  $(K_{0,s})$ , some subsequence  $(K_{1,s})$  of  $(k_{1,s})$  and some subsequence  $(K_{2,s})$  of  $(k_{2,s})$ .

It now remains to show that the family  $\{(T_{k,\lambda})_{k \in I}\}$  satisfies the assumptions of Theorem 1.2. We have already shown that the family  $\{(T_{k,\lambda}x)_{k \geq 1}\}$  satisfies the conditions (i) and (ii). We can thus deduce that the family  $\{(T_{k,\lambda}x)_{k \in I}\}$  also satisfies these conditions. Moreover, the condition (iii) is satisfied for  $\Lambda_n = \lambda \in [-n, n] \cup (\mathbb{T}, [-n, n])$ . This follows from our choice of  $\phi$  (coming from Proposition 3.8) and our choice of  $\tilde{\phi}$ . Finally, by definition of  $I$ , we also know that the condition (iv) is satisfied. We can thus apply Theorem 1.2 to the family  $\{(T_{k,\lambda})_{k \in I}\}$  and get the existence of a common hypercyclic subspace for the family  $\{\lambda D\}_{\lambda \in \mathbb{C} \setminus \{0\}} \cup \{\mu T_a\}_{a, \mu \in \mathbb{C} \setminus \{0\}}$ .  $\square$

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